arXiv:1010.3408v1 [math.RA] 17 Oct 2010

NON-COMMUTATIVE HOM-POISSON ALGEBRAS

DONALD YAU

Abstract. A Hom-type generalization of non-commutative Poisson algebras, called non-commutative Hom-Poisson algebras, are studied. They are closed under twisting by suitable self-maps. Hom-Poisson algebras, in which the Hom-associative product is commutative, are closed under tensor products. Through (de)polarization Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which has only one binary operation. Multiplicative admissible Hom-Poisson algebras are Hom-power associative.

1. Introduction

A Poisson algebra $(A, \{,\}, \mu)$ consists of a commutative associative algebra (A, μ) together with a Lie algebra structure $\{,\}$, satisfying the Leibniz identity:

$$\{\mu(x, y), z\} = \mu(\{x, z\}, y) + \mu(x, \{y, z\}).$$

Poisson algebras are used in many fields in mathematics and physics. In mathematics, Poisson algebras play a fundamental role in Poisson geometry [18], quantum groups [4, 5], and deformation of commutative associative algebras [8]. In physics, Poisson algebras are a major part of deformation quantization [11], Hamiltonian mechanics [3], and topological field theories [17]. Poisson-like structures are also used in the study of vertex operator algebras [7].

One way to generalize Poisson algebras is to omit the commutativity requirement. Such a structure is called a non-commutative Poisson algebra. When the associative product happens to be commutative, one has a Poisson algebra. Some classification results of finite dimensional non-commutative Poisson algebras can be found in [12]. One can also think of a non-commutative Poisson algebra as a special case of a Leibniz pair [6].

The purpose of this paper is to study a twisted generalization of non-commutative Poisson algebras, called non-commutative Hom-Poisson algebras. In a non-commutative Hom-Poisson algebra A, there is a linear self-map α (the twisting map) and two binary operations $\{,\}$ (the Hom-Lie bracket) and μ (the Hom-associative product). The associativity, the Jacobi identity, and the Leibniz identity in a non-commutative Poisson algebra are replaced by their Hom-type (i.e., α-twisted) analogues in a non-commutative Hom-Poisson algebra. In particular, (A, μ, α) is a Hom-associative algebra [13], and $(A, \{,\}, \alpha)$ is a Hom-Lie algebra [10]. If the twisting map is the identity map, then a non-commutative Hom-Poisson algebra reduces to a non-commutative Poisson algebra.

Most of our results are about Hom-Poisson algebras, which are non-commutative Hom-Poisson algebras in which the Hom-associative products are commutative. Hom-Poisson algebras were defined in [15] by Makhlouf and Silvestrov. It is shown in [15] that Hom-Poisson algebras play the

Date: October 19, 2010.
2000 *Mathematics Subject Classification.* 17A15,17B63.
Key words and phrases. Non-commutative Hom-Poisson algebras, admissible Hom-Poisson algebras, Hom-power associative algebra.

same role in the deformation of commutative Hom-associative algebras as Poisson algebras do in the deformation of commutative associative algebras. Other Hom-type algebras are studied in [13, 14] and [19] - [26] and the references therein.

The rest of this paper is organized as follows. In section 2 non-commutative Hom-Poisson algebras are defined. It is shown that, just like Poisson algebras, Hom-Poisson algebras are closed under tensor products in a non-trivial way (Theorem 2.9). It should be noted that Hom-Lie algebras are not closed under tensor products in any non-trivial way. In section 3 it is shown that non-commutative Hom-Poisson algebras are closed under suitable twistings by weak morphisms (Theorem 3.2). This is a unique feature for Hom-type algebras, as non-commutative Poisson algebras are not closed under such twistings. Using a special case of this result, several classes of (non-commutative) Hom-Poisson algebras are constructed.

In section 4 it is shown that Hom-Poisson algebras are equivalent to admissible Hom-Poisson algebras, each of which has one twisting map and only one binary operation. The correspondence between Hom-Poisson algebras and admissible Hom-Poisson algebras is the Hom-version of the correspondence between Poisson algebras and admissible Poisson algebras [9, 16]. In section 5 it is shown that multiplicative admissible Hom-Poisson algebras are Hom-power associative.

2. Basic properties of non-commutative Hom-Poisson algebras

In this section, we introduce (non-commutative) Hom-Poisson algebras and study tensor products of Hom-Poisson algebras.

2.1. **Notations.** We work over a fixed field $\mathbf{k}$ of characteristic 0. If V is a $\mathbf{k}$-module and $\mu\colon V^{\otimes 2} \to V$ is a bilinear map, then $\mu^{op}\colon V^{\otimes 2} \to V$ denotes the opposite map, i.e., $\mu^{op} = \mu\tau$, where $\tau\colon V^{\otimes 2} \to V^{\otimes 2}$ interchanges the two variables. For a linear self-map $\alpha\colon V \to V$, denote by α^n the n-fold composition of n copies of α, with $\alpha^0 \equiv Id$.

Let us begin with the basic definitions regarding Hom-algebras.

Definition 2.2.
1. By a **Hom-module** we mean a pair (A, α) in which A is a $\mathbf{k}$-module and $\alpha\colon A \to A$ is a linear map, called the twisting map.
2. By a **Hom-algebra** we mean a triple (A, μ, α) in which (A, α) is a Hom-module and $\mu\colon A^{\otimes 2} \to A$ is a bilinear map, called the multiplication. A Hom-algebra (A, μ, α) and the corresponding Hom-module (A, α) are often abbreviated to A.
3. A Hom-algebra (A, μ, α) is said to be **multiplicative** if $\alpha\mu = \mu\alpha^{\otimes 2}$. It is called **commutative** if $\mu = \mu^{op}$.

Unless otherwise specified, an algebra (A, μ) with $\mu\colon A^{\otimes 2} \to A$ is also regarded as a Hom-algebra (A, μ, Id) with identity twisting map. Given a Hom-algebra (A, μ, α), we often use the abbreviation

$$\mu(x, y) = xy$$

for $x, y \in A$.

Let us now recall the Hom-type generalizations of associative and Lie algebras from [13].

Definition 2.3. Let (A, μ, α) be a Hom-algebra.

(1) The **Hom-associator** $as_A \colon A^{\otimes 3} \to A$ is defined as
$$as_A = \mu(\mu \otimes \alpha - \alpha \otimes \mu). \tag{2.3.1}$$
(2) The Hom-algebra A is called a **Hom-associative algebra** if it satisfies the **Hom-associative identity**
$$as_A = 0. \tag{2.3.2}$$
(3) The **Hom-Jacobian** $J_A \colon A^{\otimes 3} \to A$ is defined as
$$J_A = \mu(\mu \otimes \alpha)(Id + \sigma + \sigma^2), \tag{2.3.3}$$
where $\sigma(x \otimes y \otimes z) = z \otimes x \otimes y$. The sum $(Id + \sigma + \sigma^2)$ is called a **cyclic sum**.
(4) The Hom-algebra A is called a **Hom-Lie algebra** if it satisfies the **Hom-Jacobi identity**
$$J_A = 0. \tag{2.3.4}$$

When the twisting map α is the identity map, the above definitions reduce to the usual definitions of the associator, an associative algebra, the Jacobian, and a Lie algebra. Examples of Hom-associative and Hom-Lie algebras can be found in [13, 19, 20]. In terms of elements $x, y, z \in A$, the Hom-associator and the Hom-Jacobian are
$$\begin{aligned} as_A(x, y, z) &= (xy)\alpha(z) - \alpha(x)(yz), \\ J_A(x, y, z) &= (xy)\alpha(z) + (zx)\alpha(y) + (yz)\alpha(x). \end{aligned}$$

Let us recall the definition of a non-commutative Poisson algebra [12].

Definition 2.4. A **non-commutative Poisson algebra** $(A, \{,\}, \mu)$ consists of

(1) a Lie algebra $(A, \{,\})$ and
(2) an associative algebra (A, μ)

such that the Leibniz identity
$$\{x, yz\} = \{x, y\}z + y\{x, z\}$$
is satisfied for all $x, y, z \in A$. A **Poisson algebra** is a non-commutative Poisson algebra $(A, \{,\}, \mu)$ in which μ is commutative. A **morphism** of non-commutative Poisson algebras is a linear map that is a morphism of the underlying Lie algebras and associative algebras.

In a non-commutative Poisson algebra $(A, \{,\}, \mu)$, the Lie bracket $\{,\}$ is called the Poisson bracket, and μ is called the associative product. The Leibniz identity says that $\{x, -\}$ is a derivation with respect to the associative product. It can be rewritten in element-free form as
$$\{,\}(Id \otimes \mu) = \mu\left(\{,\} \otimes Id + (Id \otimes \{,\})(1\ 2)\right),$$
where $(1\ 2)(x \otimes y \otimes z) = y \otimes x \otimes z$.

Hom-Poisson algebras are first introduced in [14] by Makhlouf and Silvestrov. We now define the Hom-type generalization of a non-commutative Poisson algebra.

Definition 2.5. A **non-commutative Hom-Poisson algebra** $(A, \{,\}, \mu, \alpha)$ consists of

(1) a Hom-Lie algebra $(A, \{,\}, \alpha)$ and
(2) a Hom-associative algebra (A, μ, α)

such that the **Hom-Leibniz identity**

$$\{,\}(\alpha \otimes \mu) = \mu\left(\{,\} \otimes \alpha + (\alpha \otimes \{,\})(1\ 2)\right) \tag{2.5.1}$$

is satisfied. A **Hom-Poisson algebra** is a non-commutative Hom-Poisson algebra $(A, \{,\}, \mu, \alpha)$ in which μ is commutative [14].

In a non-commutative Hom-Poisson algebra $(A, \{,\}, \mu, \alpha)$, the operations $\{,\}$ and μ are called the **Hom-Poisson bracket** and the **Hom-associative product**, respectively. In terms of elements $x, y, z \in A$, the Hom-Leibniz identity says

$$\{\alpha(x), yz\} = \{x, y\}\alpha(z) + \alpha(y)\{x, z\}, \tag{2.5.2}$$

where as usual $\mu(x, y)$ is abbreviated to xy. By the anti-symmetry of the Hom-Poisson bracket $\{,\}$, the Hom-Leibniz identity is equivalent to

$$\{xy, \alpha(z)\} = \{x, z\}\alpha(y) + \alpha(x)\{y, z\}.$$

A (non-commutative) Poisson algebra is exactly a multiplicative (non-commutative) Hom-Poisson algebra with identity twisting map.

Let us now provide some basic properties of non-commutative Hom-Poisson algebras. Every associative algebra (A, μ) has a non-commutative Poisson algebra structure in which the Poisson bracket is the commutator bracket. The following result is the Hom-type analogue of this observation.

Proposition 2.6. *Let (A, μ, α) be a Hom-associative algebra. Then*

$$A^- = (A, [,] = \mu - \mu^{op}, \mu, \alpha)$$

is a non-commutative Hom-Poisson algebra.

Proof. It is proved in [13] (Proposition 1.6) that $(A, [,], \alpha)$ is a Hom-Lie algebra. Indeed, the commutator bracket $[,]$ is obviously anti-symmetric. One can write out all twelve terms (in terms of μ) in the Hom-Jacobian of A^- and observe that their sum is zero. To check the Hom-Leibniz identity (2.5.2) for A^-, we compute as follows:

$$\begin{aligned}
&[x, y]\alpha(z) + \alpha(y)[x, z] - [\alpha(x), yz] \\
&\qquad = (xy)\alpha(z) - (yx)\alpha(z) + \alpha(y)(xz) - \alpha(y)(zx) - \alpha(x)(yz) + (yz)\alpha(x) \\
&\qquad = as_A(x, y, z) - as_A(y, x, z) + as_A(y, z, x).
\end{aligned}$$

Since $as_A = 0$, we conclude that A^- satisfies the Hom-Leibniz identity. □

Next we study tensor products of Hom-Poisson algebras. For motivation, recall that Lie algebras are not closed under tensor products. The same is true for Hom-Lie algebras. On the other hand, Poisson algebras are closed under tensor products. In the rest of this section, we show that Hom-Poisson algebras are also closed under tensor products.

We need the following preliminary result.

Lemma 2.7. *Let (A, μ, α) be a commutative Hom-associative algebra. Then the expression $(xy)\alpha(z)$ is invariant under permutations of $x, y, z \in A$. In other words, we have*

$$\mu(\mu \otimes \alpha) = \mu(\mu \otimes \alpha)\pi$$

for all the permutations π on three letters.

Proof. Pick $x, y, z \in A$. Then

$$\begin{aligned}(xy)\alpha(z) &= (yx)\alpha(z)\\ &= \alpha(y)(xz)\\ &= (xz)\alpha(y).\end{aligned}$$

So the expression $(xy)\alpha(z)$ is invariant under the transpositions $(1\ 2)$ and $(2\ 3)$. The proof is complete because these two transpositions generate the symmetric group on three letters. □

Definition 2.8. Let $(A, \{,\}, \mu, \alpha)$ be a quadruple in which (A, α) is a Hom-module and $\{,\}, \mu \colon A^{\otimes 2} \to A$ are bilinear operations. Define its **Hom-associator** as_A and **Hom-Jacobian** J_A using μ and $\{,\}$, respectively, i.e.,

$$\begin{aligned}as_A &= \mu(\mu \otimes \alpha - \alpha \otimes \mu),\\ J_A &= \{,\}(\{,\} \otimes \alpha)(Id + \sigma + \sigma^2),\end{aligned}$$

where $\sigma(x \otimes y \otimes z) = z \otimes x \otimes y$.

Now we are ready to prove that Hom-Poisson algebras are closed under tensor products.

Theorem 2.9. *Let $(A_i, \{,\}_i, \mu_i, \alpha_i)$ be Hom-Poisson algebras for $i = 1, 2$, and let $A = A_1 \otimes A_2$. Define the operations $\alpha \colon A \to A$ and $\mu, \{,\} \colon A^{\otimes 2} \to A$ by:*

$$\begin{aligned}\alpha &= \alpha_1 \otimes \alpha_2,\\ \mu(x_1 \otimes x_2, y_1 \otimes y_2) &= \mu_1(x_1, y_1) \otimes \mu_2(x_2, y_2),\\ \{x_1 \otimes x_2, y_1 \otimes y_2\} &= \{x_1, y_1\}_1 \otimes \mu_2(x_2, y_2) + \mu_1(x_1, y_1) \otimes \{x_2, y_2\}_2\end{aligned}$$

for $x_i, y_i \in A_i$. Then $(A, \{,\}, \mu, \alpha)$ is a Hom-Poisson algebra.

Proof. That (A, μ, α) is a commutative Hom-associative algebra follows from the commutativity and Hom-associativity of both μ_i. Also, the commutativity of the μ_i and the anti-symmetry of the $\{,\}_i$ imply the anti-symmetry of $\{,\}$. It remains to prove the Hom-Jacobi identity and the Hom-Leibniz identity in A.

To simplify the typography, we abbreviate μ_1, μ_2, and μ using juxtaposition and drop the subscripts in $\{,\}_i$ and α_i. Pick $x = x_1 \otimes x_2$, $y = y_1 \otimes y_2$, and $z = z_1 \otimes z_2 \in A$. Then

$$\begin{aligned}\{\{x, y\}, \alpha(z)\} &= \{\{x_1, y_1\} \otimes x_2y_2, \alpha(z_1) \otimes \alpha(z_2)\} + \{x_1y_1 \otimes \{x_2, y_2\}, \alpha(z_1) \otimes \alpha(z_2)\}\\ &= \underbrace{\{\{x_1, y_1\}, \alpha(z_1)\} \otimes (x_2y_2)\alpha(z_2)}_{a} + \underbrace{\{x_1, y_1\}\alpha(z_1) \otimes \{x_2y_2, \alpha(z_2)\}}_{b}\\ &\quad + \underbrace{\{x_1y_1, \alpha(z_1)\} \otimes \{x_2, y_2\}\alpha(z_2)}_{c} + \underbrace{(x_1y_1)\alpha(z_1) \otimes \{\{x_2, y_2\}, \alpha(z_2)\}}_{d}\end{aligned}$$

The cyclic sum over x, y, z of the term a is 0 by Lemma 2.7 and the Hom-Jacobi identity in A_1. Likewise, the cyclic sum over x, y, z of the term d is 0 by Lemma 2.7 and the Hom-Jacobi identity in A_2. It follows that the Hom-Jacobian $J_A(x, y, z)$ consists of only the cyclic sum over x, y, z of the terms b and c. Using the Hom-Leibniz identity in the A_i, the terms b and c become

$$\begin{aligned}b &= \{x_1, y_1\}\alpha(z_1) \otimes \{x_2y_2, \alpha(z_2)\}\\ &= \{x_1, y_1\}\alpha(z_1) \otimes \alpha(x_2)\{y_2, z_2\} + \{x_1, y_1\}\alpha(z_1) \otimes \{x_2, z_2\}\alpha(y_2),\\ c &= \{x_1y_1, \alpha(z_1)\} \otimes \{x_2, y_2\}\alpha(z_2)\\ &= \alpha(x_1)\{y_1, z_1\} \otimes \{x_2, y_2\}\alpha(z_2) + \{x_1, z_1\}\alpha(y_1) \otimes \{x_2, y_2\}\alpha(z_2).\end{aligned}$$

Therefore, the Hom-Jacobian of A applied to $x \otimes y \otimes z$ is:

$$\begin{aligned}
J_A(x,y,z) = {} & \underbrace{\{x_1,y_1\}\alpha(z_1) \otimes \alpha(x_2)\{y_2,z_2\}}_{b_1} + \underbrace{\{x_1,y_1\}\alpha(z_1) \otimes \{x_2,z_2\}\alpha(y_2)}_{b_2} \\
& + \underbrace{\alpha(x_1)\{y_1,z_1\} \otimes \{x_2,y_2\}\alpha(z_2)}_{c_1} + \underbrace{\{x_1,z_1\}\alpha(y_1) \otimes \{x_2,y_2\}\alpha(z_2)}_{c_2} \\
& + \underbrace{\{z_1,x_1\}\alpha(y_1) \otimes \alpha(z_2)\{x_2,y_2\}}_{b_3} + \underbrace{\{z_1,x_1\}\alpha(y_1) \otimes \{z_2,y_2\}\alpha(x_2)}_{b_4} \\
& + \underbrace{\alpha(z_1)\{x_1,y_1\} \otimes \{z_2,x_2\}\alpha(y_2)}_{c_3} + \underbrace{\{z_1,y_1\}\alpha(x_1) \otimes \{z_2,x_2\}\alpha(y_2)}_{c_4} \\
& + \underbrace{\{y_1,z_1\}\alpha(x_1) \otimes \alpha(y_2)\{z_2,x_2\}}_{b_5} + \underbrace{\{y_1,z_1\}\alpha(x_1) \otimes \{y_2,x_2\}\alpha(z_2)}_{b_6} \\
& + \underbrace{\alpha(y_1)\{z_1,x_1\} \otimes \{y_2,z_2\}\alpha(x_2)}_{c_5} + \underbrace{\{y_1,x_1\}\alpha(z_1) \otimes \{y_2,z_2\}\alpha(x_2)}_{c_6}.
\end{aligned}$$

Using the anti-symmetry of $\{,\}_i$ and the commutativity of μ_i, observe that the following sums are 0: $b_1 + c_6$, $b_2 + c_3$, $b_3 + c_2$, $b_4 + c_5$, $b_5 + c_4$, and $b_6 + c_1$. This shows that $(A, \{,\}, \alpha)$ satisfies the Hom-Jacobi identity $J_A = 0$.

Finally, we check the Hom-Leibniz identity in A. Using the Hom-associativity and the Hom-Leibniz identity in the A_i and Lemma 2.7, we compute as follows:

$$\begin{aligned}
\{xy, \alpha(z)\} &= \{x_1y_1 \otimes x_2y_2, \alpha(z_1) \otimes \alpha(z_2)\} \\
&= \{x_1y_1, \alpha(z_1)\} \otimes (x_2y_2)\alpha(z_2) + (x_1y_1)\alpha(z_1) \otimes \{x_2y_2, \alpha(z_2)\} \\
&= \alpha(x_1)\{y_1,z_1\} \otimes \alpha(x_2)(y_2z_2) + \{x_1,z_1\}\alpha(y_1) \otimes (x_2z_2)\alpha(y_2) \\
&\quad + \alpha(x_1)(y_1z_1) \otimes \alpha(x_2)\{y_2,z_2\} + (x_1z_1)\alpha(y_1) \otimes \{x_2,z_2\}\alpha(y_2) \\
&= (\alpha(x_1) \otimes \alpha(x_2))\left(\{y_1,z_1\} \otimes y_2z_2 + y_1z_1 \otimes \{y_2,z_2\}\right) \\
&\quad + \left(\{x_1,z_1\} \otimes x_2z_2 + x_1z_1 \otimes \{x_2,z_2\}\right)(\alpha(y_1) \otimes \alpha(y_2)) \\
&= \alpha(x)\{y,z\} + \{x,z\}\alpha(y).
\end{aligned}$$

This shows that A satisfies the Hom-Leibniz identity. $\square$

Setting $\alpha_i = Id_{A_i}$ in Theorem 2.9, we recover the following well-known result about Poisson algebras.

Corollary 2.10. *Let $(A_i, \{,\}_i, \mu_i)$ be Poisson algebras for $i = 1, 2$, and let $A = A_1 \otimes A_2$. Define the operations $\mu, \{,\} \colon A^{\otimes 2} \to A$ by:*

$$\begin{aligned}
\mu(x_1 \otimes x_2, y_1 \otimes y_2) &= \mu_1(x_1,y_1) \otimes \mu_2(x_2,y_2), \\
\{x_1 \otimes x_2, y_1 \otimes y_2\} &= \{x_1,y_1\}_1 \otimes \mu_2(x_2,y_2) + \mu_1(x_1,y_1) \otimes \{x_2,y_2\}_2
\end{aligned}$$

for $x_i, y_i \in A_i$. Then $(A, \{,\}, \mu)$ is a Poisson algebra.

3. Twistings of non-commutative Hom-Poisson algebras

In this section, we first observe that non-commutative Hom-Poisson algebras are closed under twisting by suitable self-maps. A special case of this observation is that non-commutative Poisson algebras give rise to multiplicative non-commutative Hom-Poisson algebras via twisting by self-morphisms (Corollary 3.4). To study whether these twistings of non-commutative Poisson algebras

actually give rise to non-commutative Poisson algebras, we introduce the concept of *rigidity* in Definition 3.5. Twistings and rigidity of some classes of Poisson algebras are then studied.

Definition 3.1. Let $(A, \{,\}, \mu, \alpha)$ be a quadruple in which (A, α) is a Hom-module and $\{,\}, \mu \colon A^{\otimes 2} \to A$ are bilinear operations.

(1) A is **multiplicative** if
$$\alpha\{,\} = \{,\}\alpha^{\otimes 2} \quad \text{and} \quad \alpha\mu = \mu\alpha^{\otimes 2}.$$
(2) Let $(B, \{,\}_B, \mu_B, \alpha_B)$ be another such quadruple. A **weak morphism** $f \colon A \to B$ is a linear map such that
$$f\{,\} = \{,\}_B f^{\otimes 2} \quad \text{and} \quad f\mu = \mu_B f^{\otimes 2}.$$
A **morphism** $f \colon A \to B$ is a weak morphism such that $f\alpha = \alpha_B f$.

Note that a quadruple $(A, \{,\}, \mu, \alpha)$ is multiplicative if and only if the twisting map $\alpha \colon A \to A$ is a morphism.

The following result says that (non-commutative) Hom-Poisson algebras are closed under twisting by self-weak morphisms.

Theorem 3.2. *Let $(A, \{,\}, \mu, \alpha)$ be a (non-commutative) Hom-Poisson algebra and $\beta \colon A \to A$ be a weak morphism. Then*
$$A_\beta = (A, \{,\}_\beta = \beta\{,\}, \mu_\beta = \beta\mu, \beta\alpha)$$
is also a (non-commutative) Hom-Poisson algebra. Moreover, if A is multiplicative and β is a morphism, then A_β is a multiplicative (non-commutative) Hom-Poisson algebra.

Proof. If μ is commutative, then clearly so is μ_β. The rest of the proof applies whether μ is commutative or not.

Since β is compatible with μ and $\{,\}$, the Hom-associators and the Hom-Jacobians of A and A_β are related as:
$$\begin{aligned}\beta^2 as_A &= (\beta\mu)\,(\beta\mu \otimes \beta\alpha - \beta\alpha \otimes \beta\mu)\\ &= as_{A_\beta}\end{aligned}$$
and
$$\begin{aligned}\beta^2 J_A &= (\beta\{,\})(\beta\{,\} \otimes \beta\alpha)(Id + \sigma + \sigma^2)\\ &= J_{A_\beta}.\end{aligned}$$
This implies that $(A, \mu_\beta, \beta\alpha)$ is a Hom-associative algebra and that $(A, \{,\}_\beta, \beta\alpha)$ is a Hom-Lie algebra. Likewise, applying β^2 to the Hom-Leibniz identity (2.5.1) in A, we obtain
$$(\beta\{,\})(\beta\alpha \otimes \beta\mu) = (\beta\mu)\,(\beta\{,\} \otimes \beta\alpha + (\beta\alpha \otimes \beta\{,\})(1\ 2))\,,$$
which is the Hom-Leibniz identity in A_β.

For the multiplicativity assertion, suppose A is multiplicative and β is a morphism. Let ν denote either μ or $\{,\}$. Then
$$\begin{aligned}(\beta\alpha)\nu_\beta &= (\beta\alpha)(\beta\nu)\\ &= \beta\nu\alpha^{\otimes 2}\beta^{\otimes 2}\\ &= \nu_\beta(\alpha\beta)^{\otimes 2}\\ &= \nu_\beta(\beta\alpha)^{\otimes 2}.\end{aligned}$$

This shows that A_β is multiplicative. □

Two special cases of Theorem 3.2 follow. The following result says that each multiplicative (non-commutative) Hom-Poisson algebra gives rise to a derived sequence of multiplicative (non-commutative) Hom-Poisson algebras.

Corollary 3.3. *Let $(A, \{,\}, \mu, \alpha)$ be a multiplicative (non-commutative) Hom-Poisson algebra. Then*

$$A^n = \left(A, \{,\}^{(n)} = \alpha^n\{,\}, \mu^{(n)} = \alpha^n\mu, \alpha^{n+1}\right)$$

is a multiplicative (non-commutative) Hom-Poisson algebra for each integer $n \geq 0$.

Proof. The multiplicativity of A implies that $\alpha^n \colon A \to A$ is a morphism. By Theorem 3.2 $A_{\alpha^n} = A^n$ is a multiplicative (non-commutative) Hom-Poisson algebra. □

The following observation is the $\alpha = Id$ special case of Theorem 3.2. It gives a twisting construction of multiplicative (non-commutative) Hom-Poisson algebras from (non-commutative) Poisson algebras. A result of this form was first given by the author in [20] for G-Hom-associative algebras. This twisting construction highlights the fact that the usual category of (non-commutative) Poisson algebras is *not* closed under twisting by self-morphisms, which, in view of Theorem 3.2, is in strong contrast with the category of (non-commutative) Hom-Poisson algebras. This is a major conceptual difference between (non-commutative) Hom-Poisson algebras and (non-commutative) Poisson algebras.

Corollary 3.4. *Let $(A, \{,\}, \mu)$ be a (non-commutative) Poisson algebra and $\beta \colon A \to A$ be a morphism. Then*

$$A_\beta = (A, \{,\}_\beta = \beta\{,\}, \mu_\beta = \beta\mu, \beta)$$

is a multiplicative (non-commutative) Hom-Poisson algebra.

In the setting of Corollary 3.4, two natural questions arise: Is the triple

$$A'_\beta = (A, \{,\}_\beta = \beta\{,\}, \mu_\beta = \beta\mu) \tag{3.4.1}$$

a (non-commutative) Poisson algebra? If so, is it isomorphic to A itself? To study these questions, we introduce the following concepts.

Definition 3.5. Let $(A, \{,\}, \mu)$ be a non-commutative Poisson algebra.

(1) Given a morphism $\beta \colon A \to A$, the triple A'_β in (3.4.1) is called the **β-twisting of A.** A **twisting of** A is a β-twisting of A for some morphism $\beta \colon A \to A$.

(2) The β-twisting A'_β of A is called **trivial** if

$$\{,\}_\beta = 0 = \mu_\beta.$$

A'_β is called **non-trivial** if either $\{,\}_\beta \neq 0$ or $\mu_\beta \neq 0$.

(3) A is called **rigid** if every twisting of A is either trivial or isomorphic to A.

The reader is cautioned not to confuse the above notion of rigidity with Gerstenhaber's [8].

Every non-commutative Poisson algebra A has at least one trivial twisting A'_β, which is obtained by taking $\beta = 0$. As we will see below, it is possible for a β-twisting to be trivial even if β is not the zero map. On the other hand, if a non-commutative Poisson algebra A has either a non-zero

Lie bracket or a non-zero associative product, then the Id_A-twisting is non-trivial. As we will see below, it is possible for a β-twisting to be isomorphic to A even if β is not the identity map.

The following result gives some basic criteria for non-rigidity.

Proposition 3.6. *Let $(A, \{,\}, \mu)$ be a non-commutative Poisson algebra. Suppose there exists a morphism $\beta\colon A \to A$ such that either:*

(1) *$\mu_\beta = \beta\mu$ is not associative or*
(2) *$\{,\}_\beta = \beta\{,\}$ does not satisfy the Jacobi identity.*

Then A is not rigid.

Proof. The β-twisting A'_β is non-trivial, since otherwise μ_β would be associative and $\{,\}_\beta$ would satisfy the Jacobi identity. For the same reason, the β-twisting A'_β cannot be isomorphic to A. □

Using Proposition 3.6 we now consider some classes of non-commutative Poisson algebras and study their (non-)rigidity.

Example 3.7 (**Free associative algebras are not rigid**). Let $A = (\mathbf{k}\langle S\rangle, \mu)$ be the free unital associative algebra on a non-empty set S. Equip A with the non-commutative Poisson algebra structure in which the Poisson bracket is the commutator bracket (Proposition 2.6). We claim that A is not rigid. By Proposition 3.6 it suffices to show that $\mu_\alpha = \alpha\mu$ is not associative for some morphism α on A. Pick an element $X \in S$, and let $\alpha\colon A \to A$ be the morphism determined for $Y \in S$ by

$$\alpha(Y) = \begin{cases} 1 + X & \text{if } Y = X, \\ Y & \text{if } Y \neq X. \end{cases}$$

For $n \geq 1$, we have

$$\alpha^n(X) = n + X.$$

The associator of μ_α applied to $(X, X, \alpha(X))$ is:

$$\begin{aligned} \mu_\alpha(\mu_\alpha(X, X), \alpha(X)) &- \mu_\alpha(X, \mu_\alpha(X, \alpha(X))) \\ &= \left(\alpha^2(X)\right)^3 - \alpha(X)\alpha^2(X)\alpha^3(X) \\ &= (2 + X)^3 - (1 + X)(2 + X)(3 + X) \\ &\neq 0. \end{aligned}$$

So μ_α is not associative, and A is not rigid. □

Example 3.8 (**Matrix algebras are not rigid**). Let $A = (M_n(\mathbf{k}), \mu)$ be the associative algebra of $n \times n$ matrices over $\mathbf{k}$ for some $n \geq 2$. Equip A with the non-commutative Poisson algebra structure in which the Poisson bracket is the commutator bracket (Proposition 2.6). We claim that A is not rigid. By Proposition 3.6 it suffices to show that $\mu_\alpha = \alpha\mu$ is not associative for some morphism α on A.

To construct such a morphism, consider the diagonal matrix

$$D = \begin{pmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The inverse of D is the same as D except that its $(1,1)$-entry is 2. There is a morphism $\alpha\colon A \to A$ given by

$$\alpha((a_{ij})) = D(a_{ij})D^{-1} = \begin{pmatrix} a_{11} & a_{12}/2 & \cdots & a_{1n}/2 \\ 2a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 2a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

From here on, a matrix $(a_{ij}) \in A$ with $a_{ij} = 0$ whenever $i \geq 3$ or $j \geq 3$ is abbreviated to its upper left 2×2 submatrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Consider such a matrix

$$X = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in A.$$

Then a quick computation shows that

$$\mu_\alpha(\mu_\alpha(X,X),\alpha(X)) = \alpha^2(X^3) = \begin{pmatrix} 3 & 1/2 \\ 8 & 1 \end{pmatrix},$$

whereas

$$\mu_\alpha(X,\mu_\alpha(X,\alpha(X))) = \alpha\left(X\alpha(X)\alpha^2(X)\right) = \begin{pmatrix} 5 & 3/8 \\ 6 & 1/4 \end{pmatrix}.$$

So μ_α is not associative, and A is not rigid. □

Example 3.9 (**Hom-Poisson algebras from linear Poisson structures**)**.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, and let $(S(\mathfrak{g}),\mu)$ be its symmetric algebra. If $\{e_i\}_{i=1}^n$ is a basis of $\mathfrak{g}$, then $S(\mathfrak{g})$ is the polynomial algebra $\mathbf{k}[e_1,\ldots,e_n]$. Suppose the structure constants for $\mathfrak{g}$ are given by

$$[e_i,e_j] = \sum_{k=1}^n c_{ij}^k e_k.$$

Then the symmetric algebra $S(\mathfrak{g})$ becomes a Poisson algebra with the Poisson bracket

$$\{F,G\} = \frac{1}{2}\sum_{i,j,k=1}^n c_{ij}^k e_k \left(\frac{\partial F}{\partial e_i}\frac{\partial G}{\partial e_j} - \frac{\partial F}{\partial e_j}\frac{\partial G}{\partial e_i}\right) \tag{3.9.1}$$

for $F,G \in S(\mathfrak{g})$. This Poisson algebra structure on $S(\mathfrak{g})$ is called the **linear Poisson structure**. Note that $\{e_i,e_j\} = [e_i,e_j]$.

Let $\alpha\colon \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. It extends to an associative algebra morphism $\alpha\colon S(\mathfrak{g}) \to S(\mathfrak{g})$ determined by

$$\alpha(F(e_1,\ldots,e_n)) = F(\alpha(e_1),\ldots,\alpha(e_n)).$$

Moreover, we claim that α respects the Poisson bracket (3.9.1), i.e., α is a morphism of Poisson algebras on $S(\mathfrak{g})$. Indeed, suppose that

$$\alpha\{F,G\} = \{\alpha(F),\alpha(G)\} \quad \text{and} \quad \alpha\{H,G\} = \{\alpha(H),\alpha(G)\}$$

for some $F,G,H \in S(\mathfrak{g})$. Then the Leibniz identity

$$\{FH,G\} = \{F,G\}H + F\{H,G\}$$

implies

$$\begin{aligned}\alpha\{FH,G\} &= \alpha(\{F,G\})\alpha(H) + \alpha(F)\alpha(\{H,G\})\\ &= \{\alpha(F),\alpha(G)\}\alpha(H) + \alpha(F)\{\alpha(H),\alpha(G)\}\\ &= \{\alpha(F)\alpha(H),\alpha(G)\}\\ &= \{\alpha(FH),\alpha(G)\}.\end{aligned}$$

Likewise, if

$$\alpha\{F,G\} = \{\alpha(F),\alpha(G)\} \quad \text{and} \quad \alpha\{F,H\} = \{\alpha(F),\alpha(H)\},$$

then the Leibniz identity

$$\{F,GH\} = \{F,G\}H + G\{F,H\}$$

implies

$$\alpha\{F,GH\} = \{\alpha(F),\alpha(GH)\}.$$

Therefore, since $S(\mathfrak{g})$ is the polynomial algebra $\mathbf{k}[e_1,\ldots,e_n]$, to check that

$$\alpha\{F,G\} = \{\alpha(F),\alpha(G)\}$$

for all $F,G \in S(\mathfrak{g})$, it suffices to show

$$\alpha\{e_i,e_j\} = \{\alpha(e_i),\alpha(e_j)\} \tag{3.9.2}$$

for all $i,j \in \{1,\ldots,n\}$. The desired identity (3.9.2) is true because $\{e_i,e_j\} = [e_i,e_j]$ and α is a Lie algebra morphism on $\mathfrak{g}$.

Therefore, given a Lie algebra morphism $\alpha\colon \mathfrak{g} \to \mathfrak{g}$, the extended map $\alpha\colon S(\mathfrak{g}) \to S(\mathfrak{g})$ is a morphism of Poisson algebras. By Corollary 3.4 there is a multiplicative Hom-Poisson algebra

$$S(\mathfrak{g})_\alpha = (S(\mathfrak{g}), \{,\}_\alpha = \alpha\{,\}, \mu_\alpha = \alpha\mu, \alpha),$$

which reduces to the linear Poisson structure $S(\mathfrak{g})$ if $\alpha = Id$. □

Example 3.10 ($S(\mathfrak{sl}_2)$ **is not rigid).** In this example, we show that the symmetric algebra $(S(\mathfrak{sl}_2),\mu)$ on the Lie algebra $\mathfrak{sl}_2$, equipped with the linear Poisson structure (3.9.1), is not rigid in the sense of Definition 3.5.

The Lie algebra $\mathfrak{sl}_2$ has a basis $\{e,f,h\}$, with respect to which the Lie bracket is given by

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

To show that $S(\mathfrak{sl}_2) = \mathbf{k}[e,f,h]$ is not rigid, consider the Lie algebra morphism $\alpha\colon \mathfrak{sl}_2 \to \mathfrak{sl}_2$ given by

$$\alpha(e) = \lambda e, \quad \alpha(f) = \lambda^{-1} f, \quad \alpha(h) = h,$$

where $\lambda \in \mathbf{k}$ is a fixed scalar with $\lambda \neq 0,1$. As in Example 3.9, denote by $\alpha\colon S(\mathfrak{sl}_2) \to S(\mathfrak{sl}_2)$ the extended map, which is a Poisson algebra morphism. By Proposition 3.6, the Poisson algebra $S(\mathfrak{sl}_2)$ is not rigid if $\mu_\alpha = \alpha\mu$ is not associative. We have

$$\begin{aligned}\mu_\alpha(\mu_\alpha(e,h),h) - \mu_\alpha(e,\mu_\alpha(h,h)) &= \alpha^2(e)\alpha^2(h)\alpha(h) - \alpha(e)\alpha^2(h)\alpha^2(h)\\ &= (\lambda^2-\lambda)eh^2,\end{aligned}$$

which is not 0 in the symmetric algebra $S(\mathfrak{sl}_2)$ because $\lambda \neq 0,1$. Therefore, μ_α is not associative, and the linear Poisson structure on $S(\mathfrak{sl}_2)$ is not rigid. □

Example 3.11 (**Hom-Poisson algebras from Poisson manifolds**). In this example, we discuss how multiplicative Hom-Poisson algebras arise from Poisson manifolds, which include symplectic manifolds and Poisson-Lie groups. The reader is referred to, e.g., [4, 18] for discussion of Poisson manifolds. The ground field in this example is the field of real numbers.

Let M be a Poisson manifold. This means that M is a smooth manifold and that the commutative associative algebra $(C^\infty(M), \mu)$ (under point-wise multiplication) of smooth $\mathbb{R}$-valued functions on M is equipped with a Poisson algebra structure

$$\{,\} \colon C^\infty(M) \times C^\infty(M) \to C^\infty(M).$$

Let N be another Poisson manifold. A **Poisson map** $\varphi \colon M \to N$ between two Poisson manifolds is a smooth map such that

$$\{f, g\}\varphi = \{f\varphi, g\varphi\}$$

for all $f, g \in C^\infty(N)$. Given such a Poisson map, the induced map

$$\varphi^* \colon C^\infty(N) \to C^\infty(M), \quad \varphi^*(f) = f\varphi$$

is a morphism of Poisson algebras.

Let $\varphi \colon M \to M$ be a Poisson map. By Corollary 3.4 the morphism $\varphi^* \colon C^\infty(M) \to C^\infty(M)$ yields a multiplicative Hom-Poisson algebra

$$C^\infty(M)_{\varphi^*} = (C^\infty(M), \{,\}_{\varphi^*} = \varphi^*\{,\}, \mu_{\varphi^*} = \varphi^*\mu, \varphi^*)\,.$$

If $\varphi = Id_M$, then $C^\infty(M)_{Id^*}$ is the original Poisson algebra $C^\infty(M)$. □

We now give sufficient conditions that guarantee that the Poisson algebra $C^\infty(M)$ of smooth functions on M is *not* rigid in the sense of Definition 3.5.

Corollary 3.12. *Let M be a Poisson manifold. Suppose there exist a Poisson map $\varphi \colon M \to M$, $x \in M$, and $f \in C^\infty(M)$ such that the matrix*

$$\Phi = \begin{pmatrix} f(\varphi^2(x)) & f(\varphi(x)) \\ f(\varphi^3(x)) & f(\varphi^2(x)) \end{pmatrix}$$

has non-zero trace and non-zero determinant. Then the Poisson algebra $C^\infty(M)$ is not rigid.

Proof. Using Proposition 3.6 and the notations in Example 3.11, it suffices to show that μ_{φ^*} is not associative. That the matrix Φ has non-vanishing trace and determinant means

$$f(\varphi^2(x)) \neq 0 \quad \text{and} \quad \bigl(f(\varphi^2(x))\bigr)^2 \neq f(\varphi(x))f(\varphi^3(x)). \tag{3.12.1}$$

Using the conditions in (3.12.1), we have:

$$\begin{aligned} \mu_{\varphi^*}\left(\mu_{\varphi^*}(f, f), f\varphi\right)(x) &= \bigl(f(\varphi^2(x))\bigr)^3 \\ &\neq f(\varphi(x))f(\varphi^2(x))f(\varphi^3(x)) \\ &= \mu_{\varphi^*}\left(f, \mu_{\varphi^*}(f, f\varphi)\right)(x). \end{aligned}$$

This shows that μ_{φ^*} is not associative, so $C^\infty(M)$ is not rigid. □

The following example illustrates how the non-rigidity criterion in Corollary 3.12 can be applied.

Example 3.13 ($C^\infty(\mathbb{R}^{2n})$ **is not rigid**)**.** The Euclidean space $\mathbb{R}^{2n}$ is a Poisson manifold with Poisson structure

$$\{f,g\} = \sum_{i,j=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_{i+n}} - \frac{\partial f}{\partial x_{i+n}} \frac{\partial g}{\partial x_i} \right)$$

for $f, g \in C^\infty(\mathbb{R}^{2n})$. Using Corollary 3.12 we show that the Poisson algebra $C^\infty(\mathbb{R}^{2n})$ is not rigid in the sense of Definition 3.5.

Consider the map $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined as

$$\varphi(a_1, \ldots, a_{2n}) = (a_1 + c_1, \ldots, a_{2n} + c_{2n}),$$

where the $c_j \in \mathbb{R}$ are fixed scalars, not all of which are zero, say, $c_i \neq 0$. The map φ is a Poisson map by Chain Rule. Let $f \in C^\infty(\mathbb{R}^{2n})$ be the function

$$f(a_1, \ldots, a_{2n}) = a_i,$$

and let 0 be the origin in $\mathbb{R}^{2n}$. For each $k \geq 0$, we have

$$\varphi^k(0) = (kc_1, \ldots, kc_{2n}).$$

So

$$f(\varphi^2(0)) = 2c_i \neq 0$$

and

$$\begin{aligned} f(\varphi^2(0))^2 - f(\varphi(0))f(\varphi^3(0)) &= (2c_i)^2 - c_i(3c_i) \\ &= c_i^2 \neq 0. \end{aligned}$$

Therefore, by Corollary 3.12 the Poisson algebra $C^\infty(\mathbb{R}^{2n})$ is not rigid. □

Example 3.14 (**Every Poisson structure on the Heisenberg algebra is rigid**)**.** Over the ground field of complex numbers, the Poisson algebra structures on the Heisenberg algebra are classified by Goze and Remm in [9] (section 2). In this example, we show that these Poisson algebras are all rigid in the sense of Definition 3.5.

Let us first recall the Goze-Remm classification of Poisson algebra structures on the Heisenberg algebra [9]. Let H be the complex Heisenberg algebra, which is the three-dimensional complex Lie algebra with a basis $\{X, Y, Z\}$ such that

$$[X, Y] = Z, \quad [X, Z] = 0 = [Y, Z]. \tag{3.14.1}$$

The isomorphism classes of Poisson algebra structures on H are divided into two families. First there is a one-parameter family of Poisson algebra structures on H,

$$\mathcal{P}_{3,1}(\zeta) = \{XY = \zeta Z\},$$

where ζ is any complex number. The notation above means that in the Poisson algebra $\mathcal{P}_{3,1}(\zeta)$, the commutative associative product satisfies

$$XY = \zeta Z = YX, \tag{3.14.2}$$

and the unspecified binary products of basis elements are all zero. The only other isomorphism class of Poisson algebra structure on H is the Poisson algebra

$$\mathcal{P}_{3,2} = \{X^2 = Z\}.$$

We aim to show that the Poisson algebras $\mathcal{P}_{3,1}(\zeta)$ and $\mathcal{P}_{3,2}$ are all rigid in the sense of Definition 3.5.

Consider the Lie algebra morphisms on the Heisenberg algebra. It follows from the Heisenberg algebra relations (3.14.1) that a linear map $\alpha\colon \mathsf{H} \to \mathsf{H}$ is a Lie algebra morphism if and only if its matrix with respect to the basis $\{X, Y, Z\}$ is

$$\alpha = \begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & b \end{pmatrix} \quad \text{with} \quad b = a_{11}a_{22} - a_{21}a_{12}, \tag{3.14.3}$$

where the a_{ij} are arbitrary complex numbers. If α is a morphism on either $\mathcal{P}_{3,1}(\zeta)$ or $\mathcal{P}_{3,2}$, then we obtain further relations among the a_{ij}.

(i) We first show that $\mathcal{P}_{3,1}(\zeta)$ is rigid. If $\alpha\colon \mathcal{P}_{3,1}(\zeta) \to \mathcal{P}_{3,1}(\zeta)$ is a Poisson algebra morphism, then applying α to the relations (3.14.2) yields

$$\zeta b = \zeta(a_{11}a_{22} + a_{21}a_{12}),$$

so

$$\zeta a_{21}a_{12} = 0. \tag{3.14.4}$$

Likewise, applying α to the relations

$$X^2 = 0 = Y^2$$

in $\mathcal{P}_{3,1}(\zeta)$ yields

$$\zeta a_{11}a_{21} = 0 = \zeta a_{12}a_{22}. \tag{3.14.5}$$

Conversely, if a Lie algebra morphism $\alpha\colon \mathsf{H} \to \mathsf{H}$ satisfies (3.14.4) and (3.14.5), then it is a Poisson algebra morphism on $\mathcal{P}_{3,1}(\zeta)$. We consider the two cases: $\zeta = 0$ and $\zeta \neq 0$.

- If $\zeta = 0$, then (3.14.4) and (3.14.5) do not impose further relations on α in (3.14.3). Since $\mathcal{P}_{3,1}(0)$ has a trivial associative product μ, every α-twisting of $\mathcal{P}_{3,1}(0)$ also has a trivial Hom-associative product μ_α. With α as in (3.14.3), the induced Hom-Lie bracket as in Corollary 3.4 is determined by

$$[X, Y]_\alpha = bZ.$$

  If $b = 0$, then $[,]_\alpha = 0$ and the α-twisting

$$\mathcal{P}_{3,1}(0)'_\alpha = (\mathcal{P}_{3,1}(0), [,]_\alpha, \mu_\alpha)$$

  of $\mathcal{P}_{3,1}(0)$ is trivial. If $b \neq 0$, then the map

$$\mathcal{P}_{3,1}(0) \to \mathcal{P}_{3,1}(0)'_\alpha \quad \text{with} \quad \begin{cases} X & \mapsto & X, \\ Y & \mapsto & Y, \\ Z & \mapsto & bZ \end{cases}$$

  is an isomorphism of Poisson algebras. In other words, the Poisson algebra $\mathcal{P}_{3,1}(0)$ is rigid.
- If $\zeta \neq 0$, then the relations (3.14.4) and (3.14.5) become

$$a_{21}a_{12} = a_{11}a_{21} = a_{12}a_{22} = 0.$$

This means that α takes one of the following three forms:

$$\alpha_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{11}a_{22} \end{pmatrix},$$

$$\alpha_2 = \begin{pmatrix} a_{11} & a_{12} & 0 \\ 0 & 0 & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \quad \text{with} \quad a_{12} \neq 0,$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix} \quad \text{with} \quad a_{21} \neq 0.$$

For $\alpha = \alpha_2$ or α_3, we have $\alpha(Z) = 0$, so $[,]_\alpha = 0 = \mu_\alpha$. This implies that the α_2-twisting and the α_3-twisting of $\mathcal{P}_{3,1}(\zeta)$ (with $\zeta \neq 0$) are both trivial.

For $\alpha = \alpha_1$, if $a_{11} = 0$ or $a_{22} = 0$, then $\alpha(Z) = 0$. This implies that $[,]_\alpha = 0 = \mu_\alpha$, so the α_1-twisting of $\mathcal{P}_{3,1}(\zeta)$ (with $\zeta \neq 0$) is trivial when $a_{11}a_{22} = 0$. On the other hand, suppose $a_{11}a_{22} \neq 0$. Since

$$\begin{aligned} \mu_\alpha(X, Y) &= \zeta a_{11}a_{22}Z = \mu_\alpha(Y, X), \\ [X, Y]_\alpha &= a_{11}a_{22}Z, \end{aligned}$$

the map

$$\mathcal{P}_{3,1}(\zeta) \to \mathcal{P}_{3,1}(\zeta)'_\alpha \quad \text{with} \quad \begin{cases} X & \mapsto X, \\ Y & \mapsto Y, \\ Z & \mapsto a_{11}a_{22}Z \end{cases}$$

is an isomorphism of Poisson algebras. We have shown that when $\zeta \neq 0$, every twisting of $\mathcal{P}_{3,1}(\zeta)$ is either trivial or isomorphic to $\mathcal{P}_{3,1}(\zeta)$ itself. Therefore, $\mathcal{P}_{3,1}(\zeta)$ is rigid.

(ii) Next we show that $\mathcal{P}_{3,2}$ is rigid. Suppose $\alpha\colon \mathcal{P}_{3,2} \to \mathcal{P}_{3,2}$ is a morphism of Poisson algebras. Applying α to the relations

$$Y^2 = 0 \quad \text{and} \quad X^2 = Z,$$

we obtain

$$a_{12} = 0 \quad \text{and} \quad a_{11}a_{22} = a_{11}^2. \tag{3.14.6}$$

Conversely, if a Lie algebra morphism $\alpha\colon \mathsf{H} \to \mathsf{H}$ satisfies (3.14.6), then it is a Poisson algebra morphism on $\mathcal{P}_{3,2}$. Therefore, the Poisson algebra morphisms α on $\mathcal{P}_{3,2}$ are

$$\alpha_4 = \begin{pmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{pmatrix}$$

and

$$\alpha_5 = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{11} & 0 \\ a_{31} & a_{32} & a_{11}^2 \end{pmatrix} \quad \text{with} \quad a_{11} \neq 0.$$

Since $\alpha_4(Z) = 0$, we have $[,]_{\alpha_4} = 0 = \mu_{\alpha_4}$, so the α_4-twisting of $\mathcal{P}_{3,2}$ is trivial.

For $\alpha = \alpha_5$, we have $\alpha(Z) = a_{11}^2 Z$ and

$$\mu_\alpha(X, X) = a_{11}^2 Z = [X, Y]_\alpha.$$

So the map

$$\mathcal{P}_{3,2} \to (\mathcal{P}_{3,2})'_\alpha \quad \text{with} \quad \begin{cases} X & \mapsto \; X, \\ Y & \mapsto \; Y, \\ Z & \mapsto \; a_{11}^2 Z \end{cases}$$

is an isomorphism of Poisson algebras. We have shown that every twisting of $\mathcal{P}_{3,2}$ is either trivial or isomorphic to $\mathcal{P}_{3,2}$ itself. Therefore, $\mathcal{P}_{3,2}$ is rigid. □

4. Admissible Hom-Poisson algebras

A Poisson algebra has two binary operations, the Lie bracket and the commutative associative product. It is shown in [9, 16] that Poisson algebras can be described using only one binary operation via the polarization-depolarization process. The purpose of this section is to extend this alternative description of Poisson algebras to Hom-Poisson algebras. In other words, we will show that a Hom-Poisson algebra can be described using only the twisting map and one binary operation.

We first define the Hom-algebras that correspond to Hom-Poisson algebras.

Definition 4.1. Let (A, μ, α) be a Hom-algebra. Then A is called an **admissible Hom-Poisson algebra** if it satisfies

$$as_A(x, y, z) = \frac{1}{3}\left\{(xz)\alpha(y) - (zx)\alpha(y) + (yz)\alpha(x) - (yx)\alpha(z)\right\} \tag{4.1.1}$$

for all $x, y, z \in A$, where as_A is the Hom-associator (2.3.1) of A.

As usual in (4.1.1) the product μ is denoted by juxtapositions of elements in A. An admissible Hom-Poisson algebra with $\alpha = Id$ is exactly an **admissible Poisson algebra** as defined in [9].

To compare Hom-Poisson algebras and admissible Hom-Poisson algebras, we need the following function, which generalizes a similar function in [16].

Definition 4.2. Let (A, μ, α) be a Hom-algebra. Define the quadruple

$$P(A) = \left(A, \{,\} = \frac{1}{2}(\mu - \mu^{op}), \bullet = \frac{1}{2}(\mu + \mu^{op}), \alpha\right), \tag{4.2.1}$$

called the **polarization** of A. We call P the **polarization function**.

The following result says that admissible Hom-Poisson algebras, and only these Hom-algebras, give rise to Hom-Poisson algebras via polarization. It is the Hom-version of [16] (Example 2).

Theorem 4.3. *Let (A, μ, α) be a Hom-algebra. Then the polarization $P(A)$ is a Hom-Poisson algebra if and only if A is an admissible Hom-Poisson algebra.*

The proof will be given below. Assuming Theorem 4.3 for the moment, first we observe that the polarization function is actually a bijection from admissible Hom-Poisson algebras to Hom-Poisson algebras. To prove this statement, we introduce the following function.

Definition 4.4. Let $(A, \{,\}, \bullet, \alpha)$ be a quadruple in which (A, α) is a Hom-module and $\{,\}, \bullet \colon A^{\otimes 2} \to A$ are binary operations. Define the Hom-algebra

$$P^-(A) = (A, \mu = \{,\} + \bullet, \alpha), \tag{4.4.1}$$

called the **depolarization** of A. We call P^- the **depolarization function**.

The following result says that there is a bijective correspondence between admissible Hom-Poisson algebras and Hom-Poisson algebras via polarization and depolarization. It is the Hom-version of [9] (Proposition 3).

Corollary 4.5. *The polarization and the depolarization functions*

$$P\colon \{\textit{admissible Hom-Poisson algebras}\} \rightleftarrows \{\textit{Hom-Poisson algebras}\}\colon P^-$$

are the inverses of each other.

Proof. If (A,μ,α) is an admissible Hom-Poisson algebra, then $P(A)$ is a Hom-Poisson algebra by Theorem 4.3. We have $P^-(P(A)) = A$ because

$$\mu = \frac{1}{2}(\mu - \mu^{op}) + \frac{1}{2}(\mu + \mu^{op}).$$

Conversely, suppose $(A, \{,\}, \bullet, \alpha)$ is a Hom-Poisson algebra. To see that $P^-(A)$ is an admissible Hom-Poisson algebra, note that the anti-symmetry of $\{,\}$ and the commutativity of $\bullet$ imply that

$$\begin{aligned}\{,\} &= \frac{1}{2}\left((\{,\} + \bullet) - (\{,\} + \bullet)^{op}\right),\\ \bullet &= \frac{1}{2}\left((\{,\} + \bullet) + (\{,\} + \bullet)^{op}\right).\end{aligned}$$

So the Hom-algebra $P^-(A)$ has the property that $P(P^-(A)) = A$, which is a Hom-Poisson algebra. It follows from Theorem 4.3 that $P^-(A)$ is an admissible Hom-Poisson algebra. Since P^-P and PP^- are both identity functions, P and P^- are the inverses of each other. □

It should be noted that both the polarization and the depolarization functions preserve multiplicativity. So the polarization of a multiplicative admissible Hom-Poisson algebra is a multiplicative Hom-Poisson algebra. Conversely, the depolarization of a multiplicative Hom-Poisson algebra is a multiplicative admissible Hom-Poisson algebra.

We now prove Theorem 4.3 with a series of Lemmas. To prove the "if" part of Theorem 4.3, we need some preliminary results. The following observation says that admissible Hom-Poisson algebras are Hom-flexible. It is the Hom-version of [9] (Proposition 4). The notion of Hom-flexibility is a Hom-type generalization of the usual definition of flexibility and was first introduced in [13].

Lemma 4.6. *Every admissible Hom-Poisson algebra (A,μ,α) is Hom-flexible, i.e.,*

$$as_A(x,y,z) + as_A(z,y,x) = 0 \tag{4.6.1}$$

for all $x,y,z \in A$.

Proof. The required identity (4.6.1) follows immediately from the defining identity (4.1.1), in which the right-hand side is anti-symmetric in x and z. □

Next we observe that in an admissible Hom-Poisson algebra the cyclic sum of the Hom-associator is trivial.

Lemma 4.7. *Let (A,μ,α) be an admissible Hom-Poisson algebra. Then*

$$S_A(x,y,z) \overset{def}{=} as_A(x,y,z) + as_A(z,x,y) + as_A(y,z,x) = 0 \tag{4.7.1}$$

for all $x,y,z \in A$.

Proof. Using the defining identity (4.1.1), we have:

$$\begin{aligned}
as_A(x,y,z) &= \frac{1}{3}\left((yz)\alpha(x) + (xz)\alpha(y) - (yx)\alpha(z) - (zx)\alpha(y)\right)\\
&= -\frac{1}{3}\left((zy)\alpha(x) - (yz)\alpha(x) + (xy)\alpha(z) - (xz)\alpha(y)\right)\\
&\quad + \frac{1}{3}\left((xy)\alpha(z) - (yx)\alpha(z) + (zy)\alpha(x) - (zx)\alpha(y)\right)\\
&= -as_A(z,x,y) + as_A(x,z,y)\\
&= -as_A(z,x,y) - as_A(y,z,x).
\end{aligned}$$

The last equality above follows from Hom-flexibility (Lemma 4.6). Therefore, we conclude that $S_A = 0$. ☐

Next we show that the polarization of an admissible Hom-Poisson algebra is commutative Hom-associative.

Lemma 4.8. *Let (A, μ, α) be an admissible Hom-Poisson algebra. Then*

$$\left(A, \bullet = \frac{1}{2}(\mu + \mu^{op}), \alpha\right)$$

is a commutative Hom-associative algebra.

Proof. It is obvious that $\bullet = (\mu + \mu^{op})/2$ is commutative. To show that the Hom-associator

$$as_{P(A)} = \bullet(\bullet \otimes \alpha - \alpha \otimes \bullet)$$

is trivial, pick $x, y, z \in A$. We write μ using juxtaposition of elements in A. Expanding $as_{P(A)}$ in terms of μ, we have:

$$\begin{aligned}
4as_{P(A)}(x,y,z) &= (xy)\alpha(z) + (yx)\alpha(z) + \alpha(z)(xy) + \alpha(z)(yx)\\
&\quad - \alpha(x)(yz) - \alpha(x)(zy) - (yz)\alpha(x) - (zy)\alpha(x)\\
&= as_A(x,y,z) - as_A(z,y,x) + (yx)\alpha(z) - (yz)\alpha(x)\\
&\quad - as_A(z,x,y) + (zx)\alpha(y) + as_A(x,z,y) - (xz)\alpha(y)
\end{aligned} \tag{4.8.1}$$

Using (4.1.1) and Hom-flexibility (Lemma 4.6), we can combine six of the eight terms above as follows:

$$\begin{aligned}
as_A(x,y,z) &- as_A(z,y,x) + (yx)\alpha(z) - (yz)\alpha(x) + (zx)\alpha(y) - (xz)\alpha(y)\\
&= as_A(x,y,z) + as_A(x,y,z) - 3as_A(x,y,z)\\
&= -as_A(x,y,z)
\end{aligned} \tag{4.8.2}$$

By Lemmas 4.6 and 4.7, the other two terms in the last line in (4.8.1) become:

$$\begin{aligned}
-as_A(z,x,y) + as_A(x,z,y) &= -as_A(z,x,y) - as_A(y,z,x)\\
&= as_A(x,y,z).
\end{aligned} \tag{4.8.3}$$

Using (4.8.2) and (4.8.3) in (4.8.1), we conclude that $as_{P(A)} = 0$. ☐

Now we observe that the polarization of an admissible Hom-Poisson algebra is a Hom-Lie algebra.

Lemma 4.9. *Let (A,μ,α) be a Hom-algebra. Then*

$$\begin{aligned}4J_{P(A)}(x,y,z) &= as_A(x,y,z) + as_A(z,x,y) + as_A(y,z,x)\\ &\quad - as_A(y,x,z) - as_A(z,y,x) - as_A(x,z,y)\end{aligned} \tag{4.9.1}$$

for all $x,y,z \in A$, where $J_{P(A)}$ is the Hom-Jacobian (2.3.3) *of the polarization $P(A)$* (4.2.1)*. Moreover, if A is an admissible Hom-Poisson algebra, then*

$$\left(A, \{,\} = \frac{1}{2}(\mu - \mu^{op}), \alpha\right)$$

is a Hom-Lie algebra.

Proof. Since $\{,\} = (\mu - \mu^{op})/2$, one can expand

$$4J_{P(A)} = 4\{,\}(\{,\} \otimes \alpha)(Id + \sigma + \sigma^2)$$

in terms of μ. The resulting twelve terms are then written in terms of the Hom-associator as_A. The result is the right-hand side of (4.9.1).

For the second assertion, first note that $\{,\}$ is clearly anti-symmetric. Next observe that the identity (4.9.1) can be rewritten as

$$4J_{P(A)}(x,y,z) = S_A(x,y,z) - S_A(y,x,z),$$

where S_A is the cyclic sum of the Hom-associator defined in (4.7.1). Since $S_A = 0$ in an admissible Hom-Poisson algebra (Lemma 4.7), we conclude that $J_{P(A)} = 0$. In other words, $(A, \{,\}, \alpha)$ satisfies the Hom-Jacobi identity. □

The following result says that the polarization of an admissible Hom-Poisson algebra satisfies the Hom-Leibniz identity (2.5.2).

Lemma 4.10. *Let (A,μ,α) be a Hom-algebra. Then the polarization $P(A)$ satisfies*

$$\begin{aligned}&4\left(\{\alpha(x), y \bullet z\} - \{x,y\} \bullet \alpha(z) - \alpha(y) \bullet \{x,z\}\right)\\ &\quad= as_A(x,y,z) + as_A(z,y,x) + as_A(x,z,y) + as_A(y,z,x)\\ &\qquad - as_A(y,x,z) - as_A(z,x,y)\end{aligned} \tag{4.10.1}$$

for all $x,y,z \in A$. Moreover, if A is an admissible Hom-Poisson algebra, then the polarization $P(A)$ satisfies the Hom-Leibniz identity.

Proof. Since $\{,\} = (\mu - \mu^{op})/2$ and $\bullet = (\mu + \mu^{op})/2$, the left-hand side of (4.10.1) can be expanded in terms of μ into twelve terms. The result can be written in terms of the Hom-associator as_A, which turns out to be the right-hand side of (4.10.1).

For the second assertion, suppose that A is an admissible Hom-Poisson algebra. Then Hom-flexibility (Lemma 4.6) implies that the right-hand side of (4.10.1) is 0. We conclude that

$$\{\alpha(x), y \bullet z\} = \{x,y\} \bullet \alpha(z) + \alpha(y) \bullet \{x,z\},$$

which is the Hom-Leibniz identity in the polarization $P(A)$. □

Next we show that only admissible Hom-Poisson algebras can give rise to Hom-Poisson algebras via polarization.

Lemma 4.11. *Let (A,μ,α) be a Hom-algebra such that the polarization $P(A)$ is a Hom-Poisson algebra. Then A is an admissible Hom-Poisson algebra.*

Proof. We need to prove the identity (4.1.1). Pick $x, y, z \in A$. We will express the Hom-associator as_A in several different forms and compare them.

On the one hand, the Hom-Jacobi identity $J_{P(A)} = 0$ and (4.9.1) imply that

$$\begin{aligned} as_A(x,y,z) &= as_A(y,x,z) - as_A(y,z,x) \\ &\quad - as_A(z,x,y) + as_A(z,y,x) + as_A(x,z,y). \end{aligned} \tag{4.11.1}$$

Moreover, the Hom-Leibniz identity in $P(A)$ and (4.10.1) imply that

$$\begin{aligned} as_A(x,y,z) &= as_A(y,x,z) - as_A(y,z,x) \\ &\quad + as_A(z,x,y) - as_A(z,y,x) - as_A(x,z,y). \end{aligned} \tag{4.11.2}$$

Adding (4.11.1) and (4.11.2) and dividing the result by 2, we obtain

$$as_A(x,y,z) = as_A(y,x,z) - as_A(y,z,x), \tag{4.11.3}$$

which we will use in a moment.

On the other hand, since $\mu = \{,\} + \bullet$, we can expand the Hom-associator as_A in terms of $\{,\}$ and $\bullet$ as follows:

$$\begin{aligned} as_A(x,y,z) &= (xy)\alpha(z) - \alpha(x)(yz) \\ &= \{\{x,y\},\alpha(z)\} + \{x \bullet y, \alpha(z)\} + \{x,y\} \bullet \alpha(z) + (x \bullet y) \bullet \alpha(z) \\ &\quad - \{\alpha(x),\{y,z\}\} - \{\alpha(x), y \bullet z\} - \alpha(x) \bullet \{y,z\} - \alpha(x) \bullet (y \bullet z) \end{aligned} \tag{4.11.4}$$

Since the polarization $P(A)$ is assumed to be a Hom-Poisson algebra, we have:

$$\begin{aligned} 0 &= as_{P(A)}(x,y,z) = (x \bullet y) \bullet \alpha(z) - \alpha(x) \bullet (y \bullet z), \\ 0 &= \{x,z\} \bullet \alpha(y) - \alpha(y) \bullet \{x,z\} \\ &= \{x \bullet y, \alpha(z)\} - \alpha(x) \bullet \{y,z\} - \{\alpha(x), y \bullet z\} + \{x,y\} \bullet \alpha(z), \\ \{\{x,z\},\alpha(y)\} &= \{\{x,y\},\alpha(z)\} - \{\alpha(x),\{y,z\}\}. \end{aligned} \tag{4.11.5}$$

Using the identities (4.11.5) in (4.11.4), we obtain:

$$\begin{aligned} 4as_A(x,y,z) &= 4\{\{x,z\},\alpha(y)\} \\ &= (xz)\alpha(y) - (zx)\alpha(y) - \alpha(y)(xz) + \alpha(y)(zx) \\ &= (xz)\alpha(y) - (zx)\alpha(y) + as_A(y,x,z) - (yx)\alpha(z) - as_A(y,z,x) + (yz)\alpha(x) \\ &= (xz)\alpha(y) - (zx)\alpha(y) + (yz)\alpha(x) - (yx)\alpha(z) + as_A(x,y,z), \end{aligned}$$

where the last equality follows from (4.11.3). Finally, subtracting $as_A(x,y,z)$ in the above calculation and dividing the result by 3, we obtain the desired identity (4.1.1). □

Proof of Theorem 4.3. If A is an admissible Hom-Poisson algebra, then Lemmas 4.8, 4.9, and 4.10 imply that the polarization $P(A)$ is a Hom-Poisson algebra. The converse is Lemma 4.11. □

5. Hom-power associativity

The purpose of this section is to show that multiplicative admissible Hom-Poisson algebras are Hom-power associative. Power associativity of admissible Poisson algebras is proved in [9] (Proposition 6).

Let us first recall the definition of a Hom-power associative algebra from [26].

Definition 5.1. Let (A, μ, α) be a Hom-algebra, $x \in A$, and n be a positive integer.

(1) The n**th Hom-power** $x^n \in A$ is defined inductively as
$$x^1 = x, \qquad x^n = x^{n-1}\alpha^{n-2}(x) \tag{5.1.1}$$
for $n \geq 2$.

(2) For positive integers i and j, define
$$x^{i,j} = \alpha^{j-1}(x^i)\alpha^{i-1}(x^j). \tag{5.1.2}$$

(3) A is called n**th Hom-power associative** if
$$x^n = x^{n-i,i} \tag{5.1.3}$$
for all $x \in A$ and $i \in \{1, \ldots, n-1\}$.

(4) A is called **Hom-power associative** if A is nth Hom-power associative for all $n \geq 2$.

By definition
$$x^2 = xx, \quad x^n = x^{n-1,1}$$
for all $n \geq 2$.

If the twisting map α is the identity map, then
$$x^n = x^{n-1}x, \quad x^{i,j} = x^i x^j,$$
and nth Hom-power associativity reduces to
$$x^n = x^{n-i}x^i \tag{5.1.4}$$
for all $x \in A$ and $i \in \{1, \ldots, n-1\}$. Therefore, Hom-powers and (nth) Hom-power associativity become Albert's right powers and (nth) power associativity [1, 2] if $\alpha = Id$. Examples of Hom-power associative algebras include multiplicative right Hom-alternative algebras and non-commutative Hom-Jordan algebras. Other results for Hom-power associative algebras can be found in [26].

By definition, power associativity involves infinitely many defining identities, namely, (5.1.4) for all n. A well-known result of Albert [1] says that an algebra (A, μ) is power associative if and only if it is third and fourth power associative, i.e., the condition (5.1.4) holds for $n = 3$ and 4. Moreover, for (5.1.4) to hold for $n = 3$ and 4, it is necessary and sufficient that
$$(xx)x = x(xx) \quad \text{and} \quad ((xx)x)x = (xx)(xx)$$
for all $x \in A$. The Hom-versions of these statements are also true. More precisely, the author proved in [26] that a multiplicative Hom-algebra (A, μ, α) is Hom-power associative if and only if it is third and fourth Hom-power associative, which in turn is equivalent to
$$x^2\alpha(x) = \alpha(x)x^2 \quad \text{and} \quad x^4 = \alpha(x^2)\alpha(x^2) \tag{5.1.5}$$
for all $x \in A$.

The following result is the Hom-version of [9] (Proposition 6).

Theorem 5.2. *Every multiplicative admissible Hom-Poisson algebra is Hom-power associative.*

Proof. As discussed above, by a result in [26] it suffices to prove the two equalities in (5.1.5). Hom-flexibility (Lemma 4.6) implies that
$$0 = as_A(x, x, x) = x^2\alpha(x) - \alpha(x)x^2,$$

which proves the first identity in (5.1.5). To prove the other equality in (5.1.5), note that Hom-flexibility implies that:

$$\begin{aligned}0 &= as_A(\alpha(x), x^2, \alpha(x))\\ &= (\alpha(x)x^2)\alpha^2(x) - \alpha^2(x)(x^2\alpha(x)).\end{aligned}$$

Together with the first identity in (5.1.5), we have:

$$\begin{aligned}x^4 &\stackrel{\text{def}}{=} (x^2\alpha(x))\alpha^2(x) = (\alpha(x)x^2)\alpha^2(x)\\ &= \alpha^2(x)(x^2\alpha(x)) = \alpha^2(x)(\alpha(x)x^2).\end{aligned} \tag{5.2.1}$$

Using multiplicativity and (5.2.1), the defining identity (4.1.1) applied to $(\alpha(x), \alpha(x), x^2)$ says that:

$$\begin{aligned}0 &= 3as_A(\alpha(x), \alpha(x), x^2) - (\alpha(x)x^2)\alpha^2(x) + (x^2\alpha(x))\alpha^2(x)\\ &\quad - (\alpha(x)x^2)\alpha^2(x) + (\alpha(x)\alpha(x))\alpha(x^2)\\ &= 3\left\{(\alpha(x)\alpha(x))\alpha(x^2) - \alpha^2(x)(\alpha(x)x^2)\right\} - x^4 + (\alpha(x)\alpha(x))\alpha(x^2)\\ &= 4\alpha(x^2)\alpha(x^2) - 4x^4.\end{aligned}$$

We have proved the second identity in (5.1.5). □

References

[1] A.A. Albert, On the power-associativity of rings, Summa Brasil. Math. 2 (1948) 21-32.
[2] A.A. Albert, Power-associative rings, Trans. Amer. Math. Soc. 64 (1948) 552-593.
[3] V.I. Arnold, Mathematical methods of classical mechanics, Grad. Texts in Math. 60, Springer, Berlin, 1978.
[4] V. Chari and A.N. Pressley, A guide to quantum groups, Cambridge Univ. Press, Cambridge, 1994.
[5] V.G. Drinfel'd, Quantum groups, in: Proc. ICM (Berkeley, 1986), p.798-820, AMS, Providence, RI, 1987.
[6] M. Flato, M. Gerstenhaber, A.A. Voronov, Cohomology and deformation of Leibniz pairs, Lett. Math. Phys. 34 (1995) 77-90.
[7] E. Frenkel and D. Ben-Zvi, Vertex algebras and algebraic curves, Math Surveys and Monographs 88, 2nd ed., AMS, Providence, RI, 2004.
[8] M. Gerstenhaber, On the deformation of rings and algebras, Ann. Math. 79 (1964) 59-103.
[9] M. Goze and E. Remm, Poisson algebras in terms of non-associative algebras, J. Alg. 320 (2008) 294-317.
[10] J.T. Hartwig, D. Larsson, and S.D. Silvestrov, Deformations of Lie algebras using σ-derivations, J. Alg. 295 (2006) 314-361.
[11] M. Kontsevich, Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003) 157-216.
[12] F. Kubo, Finite-dimensional non-commutative Poisson algebras, J. Pure Appl. Alg. 113 (1996) 307-314.
[13] A. Makhlouf and S. Silvestrov, Hom-algebra structures, J. Gen. Lie Theory Appl. 2 (2008) 51-64.
[14] A. Makhlouf and S. Silvestrov, Hom-algebras and Hom-coalgebras, J. Alg. Appl. 9 (2010) 1-37.
[15] A. Makhlouf and S. Silvestrov, Notes on formal deformations of Hom-associative and Hom-Lie algebras, to appear in Forum Math., arXiv:0712.3130v1.
[16] M. Markl and E. Remm, Algebras with one operation including Poisson and other Lie-admissible algebras, J. Alg. 299 (2006) 171-189.
[17] P. Schaller and T. Strobl, Poisson structure induced (topological) field theories, Mod. Phys. Lett. A 9 (1994) 3129-3136.
[18] I. Vaisman, Lectures on the geometry of Poisson manifolds, Birkhäuser, Basel, 1994.
[19] D. Yau, Enveloping algebras of Hom-Lie algebras, J. Gen. Lie Theory Appl. 2 (2008) 95-108.
[20] D. Yau, Hom-algebras and homology, J. Lie Theory 19 (2009) 409-421.
[21] D. Yau, The Hom-Yang-Baxter equation, Hom-Lie algebras, and quasi-triangular bialgebras, J. Phys. A 42 (2009) 165202 (12pp).
[22] D. Yau, Hom-bialgebras and comodule Hom-algebras, Int. Elect. J. Alg. 8 (2010) 45-64.
[23] D. Yau, Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras, arXiv:1002.3944.
[24] D. Yau, Hom-Novikov algebras, arXiv:0909.0726.
[25] D. Yau, The Hom-Yang-Baxter equation and Hom-Lie algebras, arXiv:0905.1887.
[26] D. Yau, Hom-power associative algebras, arXiv:1007.4118.

Department of Mathematics, The Ohio State University at Newark, 1179 University Drive, Newark, OH 43055, USA

E-mail address: dyau@math.ohio-state.edu